Note on Optimal Insurance with Search Frictions

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Abstract

This short note solves the optimal insurance problem for an agent facing search frictions in the labor market. In a model with exogenous income risks (e.g. Mirrlees, 1971), the optimal contract considers the tradeoff between insurance against income risks and the incentive to work. In the presence of search frictions, income risks become endogenous as the agent sets the reservation wage in the labor market. Providing more insurance allows the agent to take more search risks by raising the reservation wage, which enables the agent to be matched with higher-paid jobs. I show that the existence of search frictions implies that the optimal contract should provide more insurance to the agent (1) by subsidizing unemployment and (2) by flattening wage income distribution during employment across different jobs. The optimal contract derived in this note theoretically justify why the income-based repayment plan of student debt repayment evaluated by Ji (2019) can significantly outperform the fixed repayment plan.

1 Model

I build a minimalist model based on McCall (1970) to analyze the optimal insurance for an agent facing search risks in the labor market.

Consider an agent born at $t = 0$ and sequentially searches for a job. Time is discrete and there is no aggregate uncertainty. The agent maximizes lifetime utility from consumption, $E\sum_{t=1}^{\infty}\beta^t u(c(t), l(t))$ with subjective rate of time preference β. The per-period utility function, $u(c, l)$, is a function of consumption c and labor supply l.

The agent can either be unemployed or employed. Starting from $t = 1$, if the agent is unemployed, the agent receives $\theta > 0$ capturing income from home production, and wage rate offers w from an exogenous cumulative distribution function $F(w)$ in each period, which is differentiable on the support $[0, \overline{w}]$. The agent needs to decide immediately whether to accept the wage offer upon receiving it. There is no recall of past wage offers. Consumption is chosen after the realization of wage offers. If the agent rejects the offer, she continues to search. Otherwise, she gets employed at wage rate w forever.

The credit market is imperfect in the sense that savings are constrained to be non-negative, $s_t \geq 0$, for all t. The interest rate on savings is r. For simplicity, I assume $\beta(1+r) = 1$ so that the agent has no incentive to transfer wealth across periods.[1]

1.1 The Reservation Wage

Denote U as the value function of an unemployed agent, and $W(w)$ as the value function of an employed agent with wage rate w. Thus,

$$W(w) = \frac{u(wl(w), l(w))}{1-\beta}, \tag{1.1}$$

where $l(w)$ is the optimal labor supply conditional on wage rate w, satisfying

$$wu_1(c,l)|_{c=wl(w)} + u_2(c,l)|_{l=l(w)} = 0. \tag{1.2}$$

When the agent rejects the wage offer, the income in the current period is θ and the value function U can be written as

$$U = u(\theta, 0) + \beta \int_0^{\overline{w}} \max\{W(w), U\} dF(w). \tag{1.3}$$

Equation (1.1) states that the agent accepts the wage offer if it provides a higher value than unemployment. Because $W(w)$ is increasing in w, the optimal job search decision follows a cutoff strategy, and the wage offer is accepted if $w > w^*$, where w^* is the reservation wage. The agent sets w^* to maximize her utility, which happens when the value of staying unemployed is equal to the value of being employed at the reservation wage, i.e., $U = W(w^*)$:

$$u(w^*l(w^*), l(w^*)) = u(\theta, 0) + \frac{\beta}{1-\beta} \int_{w^*}^{\overline{w}} [u(wl(w), l(w)) - u(w^*l(w^*), l(w^*))] dF(w). \tag{1.4}$$

2 Optimal Contract

I now characterize the optimal income transfer contract under the assumption that the reservation wage is not contractible. I show that the existence of search risks sets up the optimal contract that also considers the level of reservation wages. The novel implication is that the

[1] When the agent is unemployed, the agent does not save because she expects future income to be higher. When the agent is employed, the agent is indifferent about savings because wage income is flat and $\beta(1+r) = 1$.

lender should provide more insurance in an economy with search frictions, because this would increase the reservation wage. Following Mirrlees (1971), I assume that labor supply l is not contractable. This is equivalent to assuming that only earnings $z = wl$ are observable but not the wage rate w.

2.1 Problem Statement

The principal designs a nonlinear income transfer contract $\alpha(z)$ conditional on earnings z subject to the zero net transfer constraint and the agent's incentive compatibility constraints on labor supply and the reservation wage.

This problem is more complicated compared to the optimal income taxation problem solved by Mirrlees (1971) as there is an additional incentive compatibility constraint on the reservation wage. As I discuss in Appendix A, if the agent accepts all wage offers arriving in the first period, the problem is mathematically the same as the one solved by Mirrlees (1971) when the government has a utilitarian social welfare function.

Because the closed-form solution of the optimal contract is not attainable, I use the perturbation approach of Saez (2001) to elucidate the economics underlying the optimal contract. In particular, I characterize the second-best contract in terms of the endogenous earnings distribution $H(z)$.[2] I use z^* to denote the reservation earnings corresponding to w^*, i.e., $z^* = w^* l(w^*)$. Since there is a one-to-one mapping from w to l, there is a one-to-one mapping from w to z. Thus I can write $l(w)$ as a function of z, i.e., $l(z)$.

Formally, the principal solves the following problem:

$$\max_{\alpha(\cdot)} \frac{H(z^*)u(\theta - \alpha(\theta), 0)}{1 - \beta H(z^*)} + \int_{z^*}^{\infty} \frac{u(x - \alpha(x), l(x))}{(1 - \beta)[1 - \beta H(z^*)]} dH(x) \tag{2.1}$$

subject to the zero net transfer constraint (i.e., the expected transfer to the agent is zero),

$$\frac{H(z^*)\alpha(\theta)}{1 - \beta H(z^*)} + \int_{z^*}^{\infty} \frac{\alpha(x)}{(1 - \beta)[1 - \beta H(z^*)]} dH(x) = 0; \tag{2.2}$$

The incentive compatibility constraint on the reservation earnings (i.e., conditional on the contract $\alpha(z)$, the agent optimally sets the reservation wage w^*, which corresponds to the

[2] The shape of $H(z)$ depends on the exogenous wage offer distribution $F(w)$ and the contract $\alpha(z)$. It is difficult to elucidate the economic intuitions in terms of $F(w)$ for a general utility function. But in the problem of Mirrlees (1971), Diamond (1998) provides an intuitive characterization for the optimal income tax using a quasi-linear utility function. I do not adopt this approach because the agent becomes risk neutral when utility is quasi-linear in consumption. As a result, search risks do not matter.

reservation earnings given by $z^* = w^* l(w^*)$),

$$u(z^*, l(z^*)) = u(\theta, 0) + \frac{\beta}{1-\beta} \int_{z^*}^{\infty} [u(z, l(z)) - u(z^*, l(z^*))] dH(z); \tag{2.3}$$

The incentive compatibility constraint on labor supply (i.e., the utility from increased consumption is equal to the disutility from increased labor supply).

Proposition 1. *Let $g(z)$ be the marginal utility of consumption at earnings z expressed in terms of the shadow value of transfer λ; $\zeta^c(z)$ be the Hicksian (compensated) elasticity of earnings at z; $\eta(z)$ be the income effect on earnings at z; ζ^{z^*} be the change of expected transfer for a percentage increase in the reservation earnings z^*; and $u_z(z)$ be the derivative of utility with respect to earnings at z.*

The marginal utility of consumption during unemployment is determined by:

$$\underbrace{H(z^*)g(\theta)}_{\text{direct utility loss}} = \underbrace{H(z^*)}_{\text{mechanical effect, (+)}} + \underbrace{\frac{(1-\beta)\lambda\zeta^{z^*}}{u_z(z^*)z^*} g(\theta)}_{\text{reservation wage effect, (-)}} . \tag{2.4}$$

The optimal income transfer contract $\alpha^{SB}(z)$ is implicitly determined by:

$$\begin{aligned}
\underbrace{\int_z^{\infty} g(x) dH(x)}_{\text{direct utility loss}} = & \underbrace{1 - H(z)}_{\text{mechanical effect, (+)}} \underbrace{- \frac{z\zeta^c(z)\alpha^{SB}(z)'}{1 - \alpha^{SB}(z)' + z\zeta^c(z)\alpha^{SB}(z)''} h(z)}_{\text{elasticity effect, (-)}} \\
& \underbrace{- \int_z^{\infty} \eta(x) \frac{\alpha^{SB}(x)'}{1 - \alpha^{SB}(x)' + x\zeta^c(x)\alpha^{SB}(x)''} dH(x)}_{\text{income effect, (+)}} \\
& + \underbrace{\frac{\beta(1-\beta)\lambda\zeta^{z^*}}{u_z(z^*)z^*} \int_z^{\infty} g(x) dH(x)}_{\text{reservation wage effect, (-)}} .
\end{aligned} \tag{2.5}$$

Equation (2.5) characterizes the optimal contract. At the optimum, the direct utility loss due to a higher marginal transfer rate in the earnings interval $(z, z + dz)$ should be balanced with the benefit of more transfer to the principal[3], which consists of four effects. First, a higher marginal transfer rate raises the transfer through a mechanical effect as the agent needs to transfer more when earnings are above z. Second, there is a negative elasticity effect as the agent would reduce labor supply when her earnings are within interval $(z, z + dz)$ due to the higher marginal transfer rate. Third, there is a positive income effect as the agent would

[3]Hence, the principal can distribution the additional transfer to other states of the agent, subject to the zero net transfer constraint.

increase labor supply when earnings are above z due to higher transfer. These three effects are also existent in the formula derived by Saez (2001). The novel effect in equation (2.5) comes from the fourth term, the reservation wage effect.

It is reasonable to assume that ζ^{z^*} is positive, which implies that expected transfer to the principal increases with the reservation wage. Given that the shadow value of transfer λ is negative, the reservation wage effect should be negative. Therefore, relative to what would be if the reservation wage were non-responsive, a lower optimal marginal transfer rate is implemented in the earnings interval in which the reservation wage effect is large. This is because when the principal designs the optimal contract, he also needs to consider the endogenous movement in the reservation wage, and to some extent, try to increase it. Setting a high marginal transfer rate at the earnings where the reservation wage is more responsive would reduce the option value of staying unemployed more and disincentivize the agent from searching for better jobs, which reduces expected transfer.

The term $\int_z^\infty g(x)dH(x)$ implies that the reservation wage effect is decreasing in z. This is because increasing the marginal transfer rate in $(z, z+dz)$ increases transfer for all earnings $x > z$. Loosely speaking, the decreasing reservation wage effect makes the transfer schedule more progressive, and the principal would have a larger incentive to equalize earnings during employment.[4]

Equation (2.4) determines the marginal utility of consumption during unemployment. Because the lowest earnings are obtained during unemployment, equation (2.4) in fact implicitly determines the intercept of the optimal contract, $\alpha^{SB}(\theta)$. If the agent's reservation wage were non-responsive, then the reservation wage effect is absent in equation (2.4). In this case, the optimal contract subsidizes unemployment so that $g(\theta) = 1$, i.e., the marginal utility of consumption during unemployment is equal to the shadow cost of transfer. This is because there is no behavioral response in labor supply during unemployment; thus it is always optimal to equalize the cost of funds to the marginal utility of consumption when the agent is unemployed. However, because the agent's reservation wage is responsive, the negative reservation wage effect incentivizes the principal to set $g(\theta) < 1$, subsidizing the agent more during unemployment, which is financed by higher transfer during employment. Intuitively, this is because providing more liquidity during unemployment increases the agent's reservation wage, which would raise expected transfer.

In sum, the discussion above indicates that in the presence of search risks, the principal has the incentive to provide more insurance both by flattening the income distribution during employment and by subsidizing the income during unemployment. Insurance provision is more valuable because consumption smoothing also indirectly raises the agent's reservation wage.

[4]Note that the argument for higher progressivity is not meant to be rigorous because it also depends on how responsive $\alpha^{SB}(x)$ is for different earnings x.

Therefore, the security design in an environment with search risks should take into account both the canonical tradeoff between insurance and the incentive to work, and importantly, the response in the reservation wage.

References

Diamond, Peter A. 1998. "Optimal Income Taxation: An Example with a U-Shaped Pattern of Optimal Marginal Tax Rates." *American Economic Review*, 88(1): 83–95.

Ji, Yan. 2019. "Job Search under Debt: Aggregate Implications of Student Loans." working paper.

McCall, J. J. 1970. "Economics of Information and Job Search." *The Quarterly Journal of Economics*, 84(1): 113 – 126.

Mirrlees, J. A. 1971. "An Exploration in the Theory of Optimum Income Taxation." *Review of Economic Studies*, 38(2): 175–208.

Saez, Emmanuel. 2001. "Using Elasticities to Derive Optimal Income Tax Rates." *Review of Economic Studies*, 68: 205–229.

Appendix

In this appendix section, I solve the optimal income transfer contract when labor supply is elastic. I first show that when there is no job search (i.e., the reservation wage is fixed at $w^* = 0$), the mathematical problem is exactly the same as Mirrlees (1971)'s problem with a utilitarian social welfare function. I then show that my problem is different due to the introduction of endogenous job search decisions. I formulate the optimal contracting problem and use the perturbation approach inspired by Saez (2001) to elucidate the economic channels.

A Without Job Search

When the reservation wage w^* is set to be 0, the agent accepts all wage offers drawn from $F(w)$ in the first period. Therefore, the agent's life-time utility conditional on receiving a wage offer w is

$$W(w) = \frac{u(w,l)}{1-\beta}, \tag{A.1}$$

where l is the labor supply that satisfies the first-order condition.

To maximize the agent's expected life-time utility, the principal chooses an optimal nonlinear transfer schedule $\alpha^{SB}(z)$, as a function of the agent's earnings $z = wl$ subject to the zero net transfer constraint. The nonlinear transfer schedule is not written on wage rates because wage rates are not observable or contractible.[5] The intercept $\alpha^{SB}(0)$ can be thought of as a lump-sum transfer that is applied to any realization of earnings. The marginal transfer rate is $\alpha^{SB}(z)'$.

This problem is exactly the same as Mirrlees (1971) if we interpret it in the following way. There is a continuum of agents with different skills w and homogeneous utility functions $\frac{u(c,l)}{1-\beta}$. They work in a static economy and optimally choose their labor supply l in the tax system. The government values a utilitarian social welfare function and optimally designs a nonlinear tax schedule $\alpha^{SB}(z)$ in terms of earnings z to maximize social welfare conditional on collecting zero tax revenue.

The problem is solved by Mirrlees (1971) by applying an optimal control approach on direct truth-telling mechanisms. The advantage of this approach comes from its rigorousness to obtain the technical conditions.[6] However, the derived formula is not useful to elucidate the economic intuitions underlying the optimal contract.

[5]If wage rates are contractible, then the first-best allocation is attainable because labor supply would not be distorted by transfer contracts. It is reasonable to assume that wage rates are unobservable because if they are observable, then labor supply is also observable from wage income. But this contradicts with the assumption made in the optimal income taxation literature.

[6]For example, in order to have the local incentive-compatibility constraint being sufficient, the problem is required to satisfy the Spence-Mirrlees single crossing condition and the monotonicity condition.

B With Job Search

Now I consider the optimal contracting problem with endogenous job search decisions as specified in section 1. The only departure from the problem of Mirrlees (1971) is that the agent chooses a reservation wage below which the wage offer is rejected. Therefore, in this problem, the types of agents in the problem of Mirrlees (1971) are restricted to a mass point with earnings θ whose probability $F(w^*)$ and a continuum of types in $[w^*, \overline{w}]$ with density $\frac{f(w)}{1-F(w^*)}$, where w^* is chosen by the agent to maximize her utility.

Facing any nonlinear contract $\alpha(z)$ in terms of earnings z, the agent makes two decisions to maximize her utility. First, the agent chooses a reservation wage w^*. Second, conditional on accepting the wage offer w, the agent chooses her labor supply l. Therefore, the resulting distribution of earnings $H(z)$ depends both on the exogenous wage offer distribution $F(w)$ and the transfer schedule $\alpha(z)$.

Below, I use a perturbation approach inspired by Saez (2001) to characterize the shape of the optimal transfer contract $\alpha^{SB}(z)$. For tractability, I make the following assumptions.

Assumption 1. *Earnings z and utility $u(z - \alpha^{SB}(z), l^{SB}(z))$ weakly increase with wage rates w under the optimal contract $\alpha^{SB}(z)$.*

Assumption 2. *The optimal contract $\alpha^{SB}(z)$ is twice differentiable for all z.*

Assumption 1 is saying that the agent earns more and enjoys higher utility at jobs with higher wage rates. This is intuitively reasonable given that the monotonicity condition in the mechanism design problem of Mirrlees (1971) requires net earnings $z - \alpha^{SB}(z)$ to be weakly increasing in w. This assumption ensures that there is an injective function under the optimal contract $\alpha^{SB}(z)$, $w \mapsto z = q(w)$. Thus I denote z^* as the earnings corresponding to the reservation wage offer w^*, i.e., $z^* = q(w^*)$.

Assumption 2 comes from Saez (2001). This assumption has additional meaning in the problem I solve because it also restricts the specification of contract off the equilibrium, i.e., for $z \in (\theta, z^*)$ (see Figure B.1 for an illustration). In general, because the agent rejects the wage offer whenever the resulting earnings are below z^*, there exist infinite numbers of optimal contracts in my problem, and some of them could have a discontinuous jump at z^*.[7] This assumption ensures that the reservation wage is derived from a first-order condition instead of being a corner solution. That is, when the reservation wage is slightly changed, the change in the agent's welfare is of second order.

[7] For example, given the optimal contract $\alpha^{SB}(z)$. We can specify $\tilde{\alpha}^{SB}(z)$ such that $\tilde{\alpha}^{SB}(z) = \alpha^{SB}(z)\ for\ z \geq z^*$ and $z - \tilde{\alpha}^{SB}(z) = \theta - \alpha^{SB}(\theta)\ for\ z < z^*$. Under $\tilde{\alpha}^{SB}(z)$, the net earnings are flat up to the reservation earnings z^*, and there is a discontinuous jump in net earnings at z^*. The contract $\tilde{\alpha}^{SB}(z)$ is incentive compatible because the agent has no incentive to change her reservation earnings z^* as reducing this lowers her utility more than what would be under $\alpha^{SB}(z)$. Moreover, $\tilde{\alpha}^{SB}(z)$ also satisfies the zero net transfer constraint so it is an optimal contract.

Denote $\lambda < 0$ as the Lagrangian multiplier associated with the zero net transfer constraint,

$$\frac{H(z^*)}{1-\beta H(z^*)}\alpha^{SB}(\theta) + \frac{1}{(1-\beta)[1-\beta H(z^*)]}\int_{z^*}^{\infty}\alpha^{SB}(z)dH(z) = 0. \tag{B.1}$$

The multiplier λ is also the shadow value measuring the change in the agent's utility when some income is transferred to the principal marginally.[8] Denote $g(z) > 0$ as the marginal value of consumption for the agent with earnings z under the optimal contract, expressed in terms of the shadow cost of transfer $(-\lambda)$, i.e.,

$$g(z) = \frac{u_1(z - \alpha^{SB}(z), l^{SB}(z))}{-\lambda}, \tag{B.2}$$

where $l^{SB}(z)$ corresponds to the labor supply at earnings z under the optimal contract $\alpha^{SB}(z)$.

I follow Saez (2001) and consider a small perturbation around the optimal transfer schedule $\alpha^{SB}(z)$. Suppose that the marginal transfer rate is increased by $d\alpha$ for earnings between z and $z + dz$, where $z \geq z^*$ (see Figure B.2). This would generate the following effects on expected transfer R, defined as:

$$R = \frac{H(z^*)}{1-\beta H(z^*)}\alpha^{SB}(\theta) + \frac{1}{(1-\beta)[1-\beta H(z^*)]}\int_{z^*}^{\infty}\alpha^{SB}(z)dH(z). \tag{B.3}$$

B.1 Various Effects on Expected Repayment

Mechanical effect The agent pays $d\alpha dz$ more when her earnings are above z, with probability $1 - H(z)$. Thus expected transfer increases by

$$M = \frac{1 - H(z)}{(1-\beta)[1-\beta H(z^*)]}d\alpha dz. \tag{B.4}$$

Elasticity effect The increase in the marginal transfer rate distorts labor supply when the agent's earnings are between z and $z + dz$, which consequently affects expected transfer. The change in earnings is caused by two effects. First, there is a direct effect due to the increase in $d\alpha$. Second, there is an indirect effect as the agent would face a different marginal transfer rate when her earnings are changed by the direct effect.

As noted by Saez (2001), the direct effect can be decomposed into two parts: an overall uncompensated increase in the marginal rate and an overall increase in virtual income. Therefore, the relevant one that determines the behavioral response is the Hicksian (compensated)

[8] The negative of λ corresponds to the social value of public funds defined by Saez (2001).

elasticity of earnings, which is defined as

$$\zeta^c(z) = \frac{1-\alpha^{SB}(z)'}{z} \frac{\partial z}{\partial(1-\alpha^{SB}(z)')}\bigg|_u. \tag{B.5}$$

Suppose that the two effects result in an earnings change by Δ, then the direct effect is $-\zeta^c(z)z\frac{d\alpha}{1-\alpha^{SB}(z)'}$, and the indirect effect is $-\zeta^c(z)z\frac{\Delta\alpha^{SB}(z)''}{1-\alpha^{SB}(z)'}$. Hence,

$$\Delta = -\zeta^c(z)z\frac{d\alpha}{1-\alpha^{SB}(z)'} - \zeta^c(z)z\frac{\Delta\alpha^{SB}(z)''}{1-\alpha^{SB}(z)'}. \tag{B.6}$$

This implies that

$$\Delta = -\zeta^c(z)z\frac{d\alpha}{1-\alpha^{SB}(z)'+\zeta^c(z)z\alpha^{SB}(z)''}. \tag{B.7}$$

Following Saez (2001), I assume that $1-\alpha^{SB}(z)'+\zeta^c(z)z\alpha^{SB}(z)'' > 0$ so that bunching of types does not occur. The elasticity effect on expected transfer is

$$\begin{aligned} E &= \frac{\Delta\alpha^{SB}(z)'h(z)dz}{(1-\beta)[1-\beta H(z^*)]} \\ &= -\frac{\zeta^c(z)z\alpha^{SB}(z)'}{1-\alpha^{SB}(z)'+\zeta^c(z)z\alpha^{SB}(z)''}\frac{h(z)}{(1-\beta)[1-\beta H(z^*)]}d\alpha dz. \end{aligned} \tag{B.8}$$

Income effect If the agent accepts a wage offer generating earnings above $z+dz$, her earnings are reduced by $d\alpha dz$ due to the higher marginal transfer rate between z and $z+dz$. This would generate an income effect that induces the agent to work more. As a result, for any $x > z+dz$, earnings increase by $\Delta(x)$, which in turn increases expected transfer. The earnings response $\Delta(x)$ is due to two effects. First, there is a direct effect due to the increase in marginal rate $d\alpha$ between z and $z+dz$. Second, there is an indirect effect due to the change in marginal rates caused by the shift in earnings.

Let $\eta(z) \leq 0$ denote the income effect and $\zeta^u(z)$ denote the Marshallian (uncompensated) elasticity of earnings at earnings z, thus the income effect is derived by the Slutsky equation,

$$\zeta^u(z) = \frac{1-\alpha^{SB}(z)'}{z}\frac{\partial z}{\partial(1-\alpha^{SB}(z)')}; \tag{B.9}$$

$$\eta(z) = \zeta^u(z) - \zeta^c(z). \tag{B.10}$$

Therefore, the direct effect is $-\frac{\eta(x)d\alpha dz}{1-\alpha^{SB}(x)'}$ and the indirect effect is $-\zeta^c(x)x\frac{\alpha^{SB}(x)''\Delta(x)}{1-\alpha^{SB}(x)'}$, and the change in earnings is

$$\Delta(x) = -\frac{\eta(x)d\alpha dz}{1-\alpha^{SB}(x)'} - \zeta^c(x)x\frac{\alpha^{SB}(x)''\Delta(x)}{1-\alpha^{SB}(x)'}, \tag{B.11}$$

which implies

$$\Delta(x) = -\eta(x)\frac{d\alpha dz}{1-\alpha^{SB}(x)' + x\zeta^c(x)\alpha^{SB}(x)''}. \tag{B.12}$$

The total income effect on expected transfer is

$$I = -\frac{d\alpha dz}{(1-\beta)[1-\beta H(z^*)]}\int_z^\infty \eta(x)\frac{\alpha^{SB}(x)'}{1-\alpha^{SB}(x)' + x\zeta^c(x)\alpha^{SB}(x)''}h(x)dx. \tag{B.13}$$

Reservation wage effect There is a fourth effect on expected transfer due to the change in reservation earnings, which is not in the problem of Mirrlees (1971). The reservation earnings are determined by the following indifference equation:

$$\frac{u(z^* - \alpha^{SB}(z^*), l^{SB}(z^*))}{1-\beta} = \frac{u(\theta - \alpha^{SB}(\theta), 0)}{1-\beta H(z^*)} + \frac{\beta}{(1-\beta)[1-\beta H(z^*)]}\int_{z^*}^\infty u(x-\alpha^{SB}(x), l^{SB}(x))dH(x), \tag{B.14}$$

where the LHS of this equation represents the value of being employed at the reservation earnings z^*, and the RHS represents the value of staying unemployed. Assumption 2 ensures that the reservation earnings also satisfy the first-order condition. Rearranging it:

$$1 = \frac{\beta}{1-\beta}\int_{z^*}^\infty \frac{u(x-\alpha^{SB}(x), l^{SB}(x)) - u(z^* - \alpha^{SB}(z^*), l^{SB}(z^*))}{u(z^* - \alpha^{SB}(z^*), l^{SB}(z^*)) - u(\theta - \alpha^{SB}(\theta), 0)}dH(x). \tag{B.15}$$

Assumption 1 ensures that the integrand is non-negative and decreasing in z^*. The integration is executed from z^* to infinity, thus the RHS of equation (B.15) decreases with z^*. The increase in the marginal transfer rate $d\alpha$ between z and $z + dz$ reduces $u(x - \alpha^{SB}(x), l^{SB}(x))$ for all $x > z$, thus lowering the RHS of equation (B.15). This implies that the reservation earnings z^* would decrease.

For $x > z$, the change $d\alpha$ would change $u(x - \alpha^{SB}(x), l^{SB}(x))$ by

$$\begin{aligned} du(x) &= -u_1(x - \alpha^{SB}(x), l^{SB}(x))d\alpha dz \\ &= g(x)\lambda d\alpha dz. \end{aligned} \tag{B.16}$$

Note that the elasticity effect and the income effect discussed above indicate that labor supply $l^{SB}(x)$ would also change due to the change $d\alpha$, but the Envelope Theorem implies that such a change does not have a first-order effect on utility. Differentiating equation (B.14) and substituting (B.16), we obtain

$$dz^* = d\alpha dz \frac{\beta\lambda}{[1-\beta H(z^*)]u_z(z^*)} \int_z^\infty g(x)dH(x), \tag{B.17}$$

where $u_z(z) = \frac{du(z-\alpha^{SB}(z), l^{SB}(z))}{dz}$ denotes the marginal change in utility due to a marginal change in earnings at z under the optimal contract $\alpha^{SB}(z)$.

The change in reservation earnings dz^* does not affect the agent's welfare due to the envelope condition from Assumption 2. However, it affects expected transfer R determined by equation (B.3). Define ζ^{z^*} as the change of expected transfer with respect to a percentage change in reservation earnings,

$$\zeta^{z^*} = \frac{\partial R}{\partial z^*} z^*. \tag{B.18}$$

Differentiating (B.3), we obtain

$$\begin{aligned}
\zeta^{z^*} &= \frac{\beta R + \alpha^{SB}(\theta) - \frac{\alpha^{SB}(z^*)}{1-\beta}}{1-\beta H(z^*)} z^* h(z^*) \\
&= \frac{\alpha^{SB}(\theta) - \frac{\alpha^{SB}(z^*)}{1-\beta}}{1-\beta H(z^*)} z^* h(z^*),
\end{aligned} \tag{B.19}$$

where the second equation is obtained by substituting $R = 0$.

In general, ζ^{z^*} could be positive or negative. We can show that $\zeta^{z^*} > 0$ for empirically reasonable elasticities of labor supply. Therefore, higher reservation earnings increase expected transfer to the principal. Using equations (B.17) and (B.19), we obtain the reservation wage effect on expected transfer:

$$\begin{aligned}
RW &= \frac{dz^*}{z^*} \zeta^{z^*} \\
&= d\alpha dz \frac{\beta\lambda\zeta^{z^*}}{[1-\beta H(z^*)]u_z(z^*)z^*} \int_z^\infty g(x)dH(x).
\end{aligned} \tag{B.20}$$

B.2 Deriving the Optimal Contract During Employment

The small perturbation around the optimal contract should have no first-order effect on the agent's utility. Therefore, the sum of the four effects, M, E, I, and RW, multiplied by the shadow cost of transfer ($-\lambda$) should be equal to the agent's expected utility loss when earnings are above z. The agent's utility under $\alpha^{SB}(z)$ is

$$\text{Welfare}_{SB} = \frac{H(z^*)u(\theta - \alpha^{SB}(\theta), 0)}{1 - \beta H(z^*)} + \int_{z^*}^{\infty} \frac{u(x - \alpha^{SB}(x), l^{SB}(x))}{(1-\beta)[1 - \beta H(z^*)]} dH(x) \tag{B.21}$$

The expected utility loss is

$$\begin{aligned} WL &= d\alpha dz \int_{z}^{\infty} \frac{u_1(x - \alpha^{SB}(x), l^{SB}(x))}{(1-\beta)[1 - \beta H(z^*)]} dH(x) \\ &= d\alpha dz \int_{z}^{\infty} \frac{-\lambda g(x)}{(1-\beta)[1 - \beta H(z^*)]} dH(x). \end{aligned} \tag{B.22}$$

Again, the Envelope Theorem implies that the change in labor supply has a second-order effect on utility. At the optimum,

$$WL = -\lambda(M + E + I + RW), \tag{B.23}$$

which implies

$$\begin{aligned} \underbrace{\int_{z}^{\infty} g(x) dH(x)}_{\text{direct utility loss}} = \underbrace{1 - H(z)}_{\text{mechanical effect}} & \underbrace{- \frac{z\zeta^c(z)\alpha^{SB}(z)'}{1 - \alpha^{SB}(z)' + z\zeta^c(z)\alpha^{SB}(z)''} h(z)}_{\text{elasticity effect}} \\ & \underbrace{- \int_{z}^{\infty} \eta(x) \frac{\alpha^{SB}(x)'}{1 - \alpha^{SB}(x)' + x\zeta^c(x)\alpha^{SB}(x)''} dH(x)}_{\text{income effect}} \\ & + \underbrace{\frac{\beta(1-\beta)\lambda\zeta^{z^*}}{u_z(z^*)z^*} \int_{z}^{\infty} g(x) dH(x)}_{\text{reservation wage effect}}. \end{aligned} \tag{B.24}$$

This equation implicitly determines the optimal contract $\alpha^{SB}(z)$. It is different from the one derived by Saez (2001) due to the existence of the reservation wage effect. As a result, it does not admit an explicit solution for $\alpha^{SB}(z)$ because ζ^{z^*} is a function of $\alpha^{SB}(z)$.

To gain some intuitions, consider the case with inelastic labor supply, which implies that there is no elasticity effect or income effect in equation (B.24). If there are no endogenous search decisions, the reservation wage effect is also absent. Then the optimal contract requires $\int_z^\infty g(x)dH(x) = 1 - H(z)$ for all $z > z^*$. This happens only when $g(z) = 1, \forall z > z^*$, suggesting perfect insurance against earnings risks.

When there are search risks, the direct utility loss is equal to the sum of the mechanical effect and the reservation wage effect. If the agent is provided with perfect insurance, $g(z) = 1$, then the marginal utility does not change when different earnings offers are accepted. This implies that the term $u_z(z^*)$ in the reservation wage effect is equal to zero. In this case, for the reservation wage effect to be well defined, it is required that $\zeta^{z^*} = 0$, which happens when the reservation earnings z^* is set to maximize expected transfer.

Note that the principal can set the reservation wage to maximize expected transfer precisely because the agent with inelastic labor supply is indifferent among different reservation wages when being perfectly insured. Hence, any reservation wage is incentive compatible. This simple discussion with inelastic labor supply highlights the role of reservation wages in optimal contract design: in the context of elastic labor supply, the optimal contract not only cares about the tradeoff between efficiency (incentive to work) and insurance, but also to some extent, uses the reservation wage to increase expected transfer in order to have a smaller distortion on efficiency.

Equation (B.24) characterizes the formula that implicitly determines the optimal marginal transfer rate during employment. In the following, I derive the optimal contract during unemployment.

B.3 Deriving the Optimal Contract During Unemployment

Suppose that transfer is increased by $d\alpha$ during unemployment, which is achieved by smoothly perturbing the transfer schedule below z^* (see Figure B.3) so that Assumption 2 is still satisfied. This is going to have a mechanical effect and a reservation wage effect on expected transfer.

The mechanical effect is given by

$$M = \frac{H(z^*)}{1 - \beta H(z^*)} d\alpha, \tag{B.25}$$

which captures the fact that the agent transfers more to the principal during unemployment. Similar to equation (B.16), for earnings θ, the increase in transfer reduces utility during unemployment by

$$du(\theta) = -u_1(\theta - \alpha^{SB}(\theta), 0) d\alpha = g(\theta)\lambda d\alpha. \tag{B.26}$$

The reservation earnings are determined by equation (B.14). Differentiating this equation and substituting (B.26) yields:

$$dz^* = d\alpha \frac{(1-\beta)\lambda g(\theta)}{[1 - \beta H(z^*)] u_z(z^*)}. \tag{B.27}$$

Thus the reservation wage effect is

$$\begin{aligned} RW &= \frac{dz^*}{z^*}\zeta^{z^*} \\ &= d\alpha \frac{(1-\beta)\lambda g(\theta)\zeta^{z^*}}{[1-\beta H(z^*)]u_z(z^*)z^*}. \end{aligned} \tag{B.28}$$

According to equation (B.21), this perturbation generates a direct utility loss:

$$\begin{aligned} WL &= \frac{H(z^*)}{1-\beta H(z^*)} u_1(\theta - \alpha^{SB}(\theta), 0) d\alpha \\ &= -\frac{H(z^*)}{1-\beta H(z^*)} g(\theta)\lambda d\alpha. \end{aligned} \tag{B.29}$$

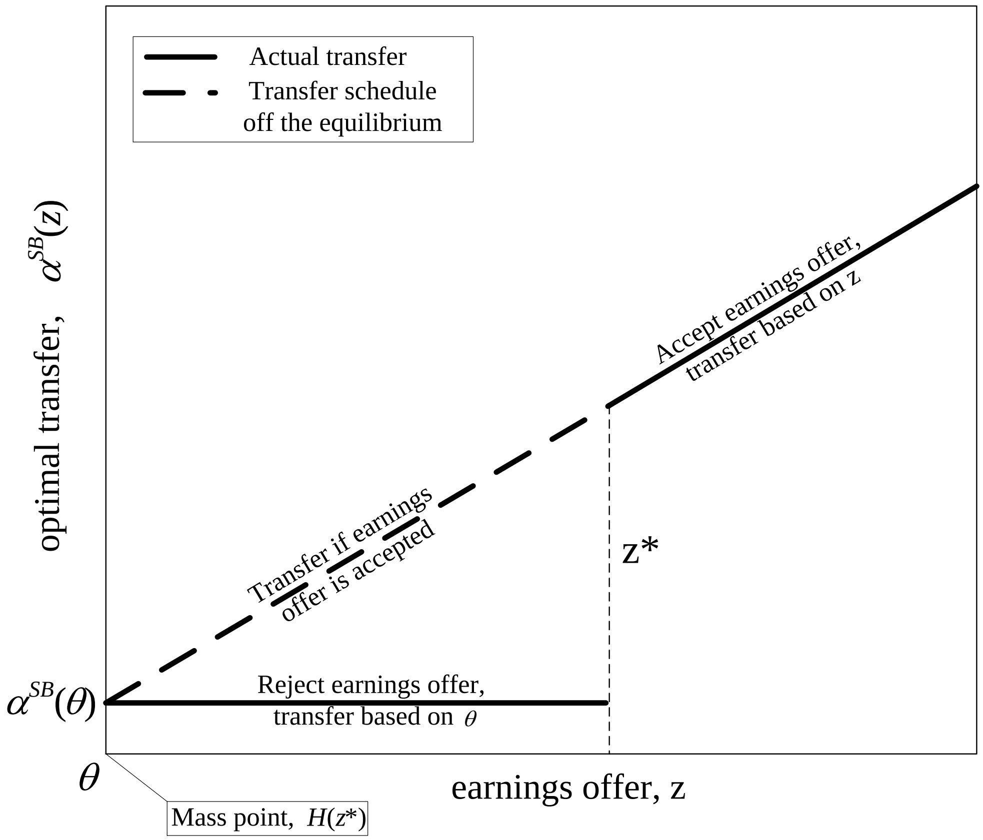
At the optimum,

$$WL = -\lambda(M + RW), \tag{B.30}$$

which yields

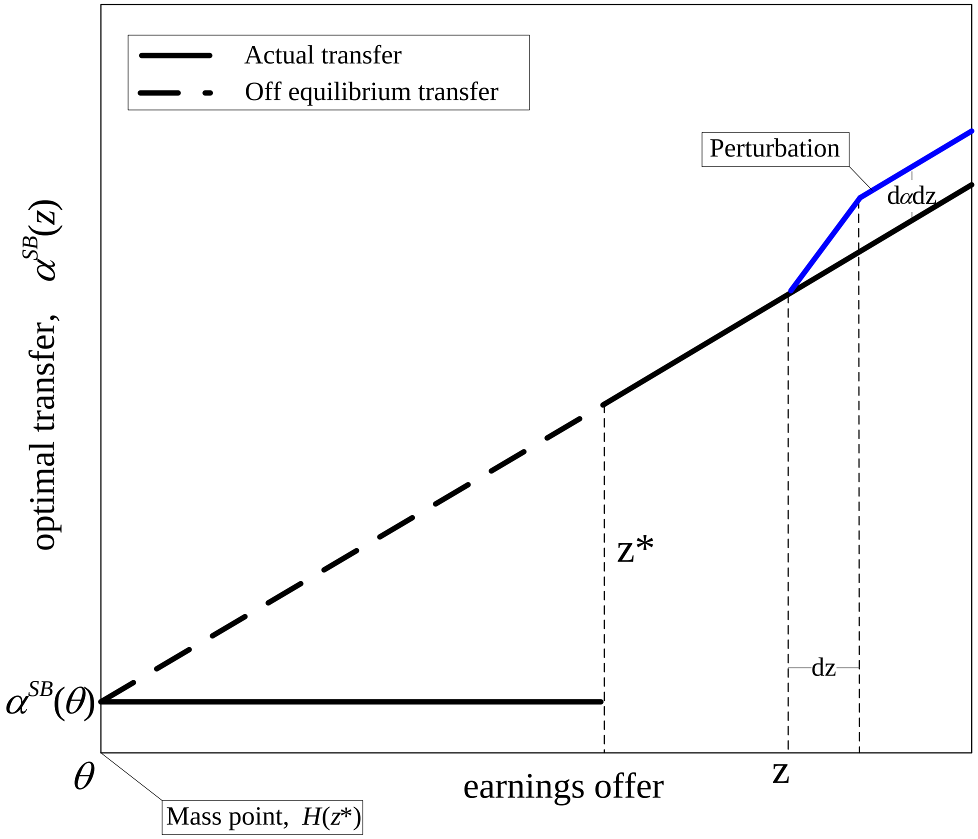
$$\underbrace{H(z^*)g(\theta)}_{\text{direct utility loss}} = \underbrace{H(z^*)}_{\text{mechanical effect}} + \underbrace{\frac{(1-\beta)\lambda\zeta^{z^*}}{u_z(z^*)z^*}g(\theta)}_{\text{reservation wage effect}}. \tag{B.31}$$

If the reservation earnings are fixed, then the reservation wage effect is absent in equation (B.31). In this case, the optimal contract subsidizes unemployment such that $g(\theta) = 1$, i.e., to the point where the marginal utility of consumption during unemployment is equal to the shadow cost of transfer. This is because there is no behavioral response during unemployment, thus it is always optimal to equalize the cost of fund to the marginal utility of consumption when the agent is unemployed. When there is a negative reservation wage effect, the optimal contract sets $g(\theta) < 1$, indicating that the principal subsidizes the agent more during unemployment. Intuitively, this is because providing more liquidity to unemployment incentivizes the agent to increase her reservation wage and search longer, which would raise expected transfer.



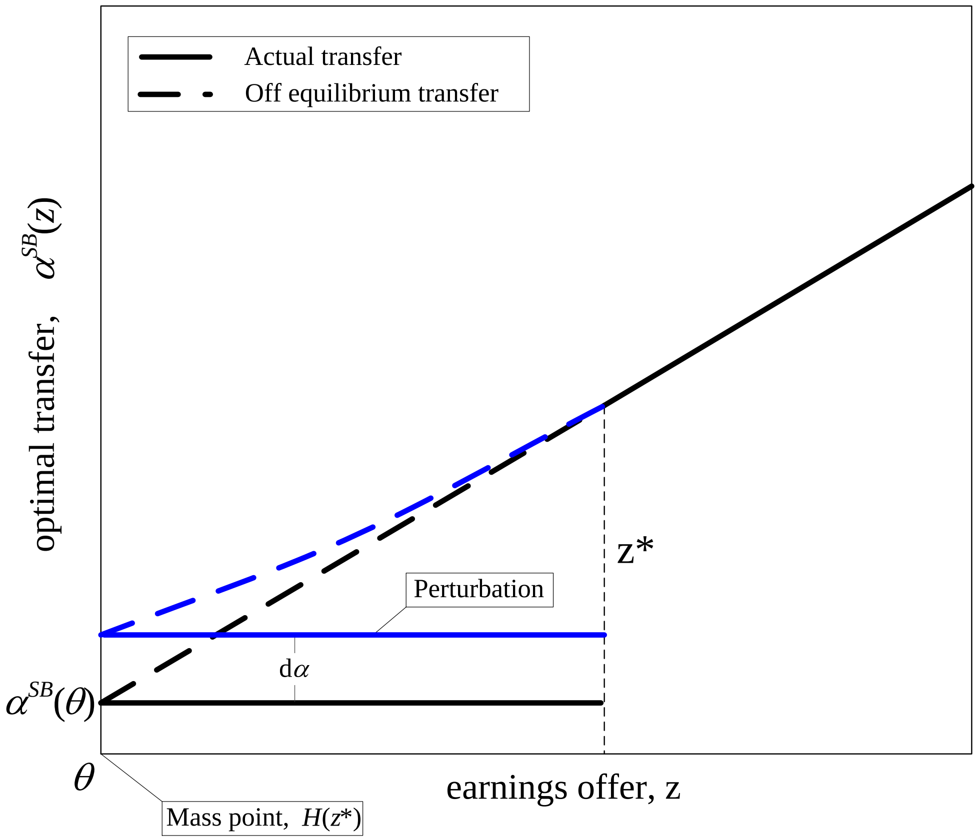
Note: This figure illustrates the off-equilibrium transfer schedule. When the wage offer generates earnings z above the reservation earnings z^*, the agent accepts the offer and receives earnings z. The amount of transfer is specified by the upward-sloping solid line for $z \geq z^*$. When the wage offer generates earnings $z < z^*$, the agent rejects the offer and receives home production θ. Therefore, for any earnings offer $z < z^*$ that the agent draws, the actual amount of transfer is equal to $\alpha^{SB}(\theta)$, based on home production. The transfer schedule off the equilibrium (dashed line) specifies the amount of transfer if the agent accepts the wage offers generating earnings between θ and z^*.

Figure B.1: An Illustration of the off-equilibrium transfer.



Note: This figure illustrates the local perturbation in marginal transfer rate considered in my analysis. The marginal transfer rate is increased by $d\alpha$ for earnings between z and $z + dz$.

Figure B.2: Local marginal transfer rate perturbation at $z > z^*$.



Note: This figure illustrates the local perturbation in transfer during unemployment in my analysis. The amount of transfer during unemployment (intercept, $\alpha^{SB}(\theta)$) is increased by $d\alpha$. The off-equilibrium schedule between θ and z^* is altered smoothly so that Assumption 2 is still satisfied.

Figure B.3: Transfer rate perturbation for the unemployed.